Vehicle Dynamics and Simulation

Differential Equations and Numerical Integration

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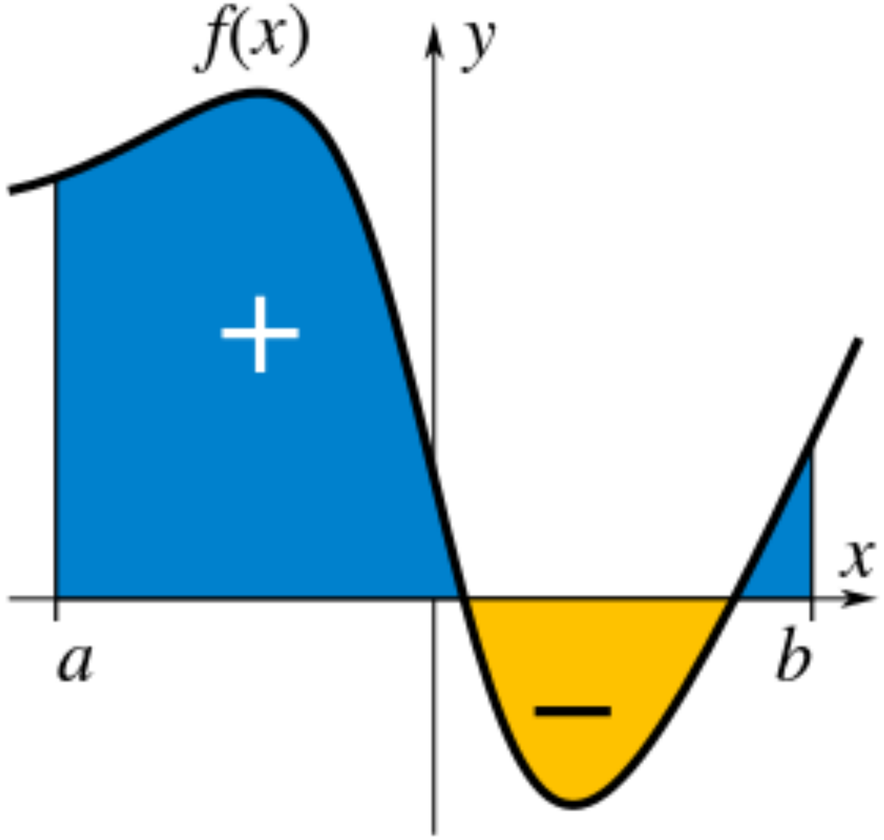
Overview

- Differential equations
- Model generation examples
- Numerical integration



Dynamic Systems Modelling: Differential Equations

- Dynamic numerical models often make use of differential equations, for example;

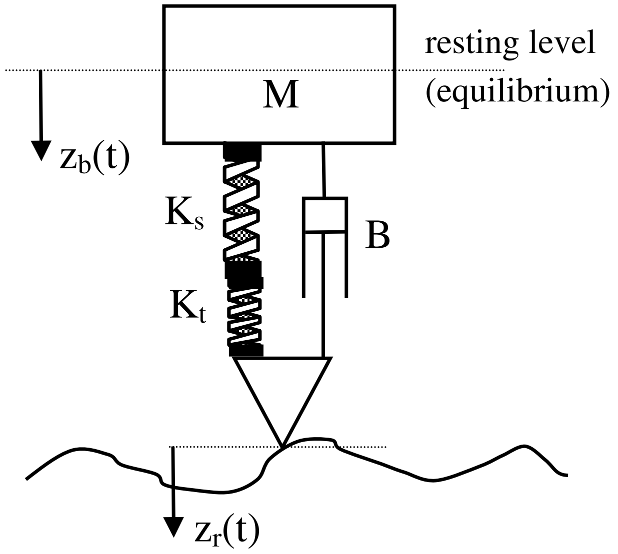
$$y = 3x^2 + 2 \qquad \frac{dy}{dx} = 6x \qquad \int y = x^3 + 2x + c$$

- A simple model may look something like this;

$$M\frac{dv}{dt} = F - \tfrac{1}{2}\rho A C_d v^2$$

How does velocity change over time, if you apply constant engine torque (and hence F *in the model above) to a car ? What are the initial and final values of* $\frac{dv}{dt}$ *and* v *?*

Modelling Example 1



Sign convention: +ve direction indicated by arrow heads

- Equations;

Equivalent Spring stiffness; $\frac{1}{K} = \frac{1}{K_s} + \frac{1}{K_t}$

$$F_s = K(z_b - z_r) + B_s(\dot{z}_b - \dot{z}_r)$$

$$\Sigma F = ma$$

$$-F_s = M\ddot{z}_b$$

$$M\ddot{z}_b = K(z_r - z_b) + B_s(\dot{z}_r - \dot{z}_b) \quad (1)$$

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Modelling Example 1

- To find a solution we need to express the Equation (1) as a system of first order equations by choosing the system 'states' correctly. Note that this is an arbitrary definition and many other choices are possible.

$$x_1 = z_b$$
$$x_2 = \dot{z}_b$$
$$x_3 = z_r$$
$$u = \dot{z}_r$$

- Making the substitutions into Equation (1)

$$M\dot{x}_2 = K(x_3 - x_1) + B_s(u - x_2)$$

Modelling Example 1

- Rearranging in terms of the states, x_1, x_2 and x_3;

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{K}{M}(x_3 - x_1) + \frac{B_s}{M}(u - x_2)$$

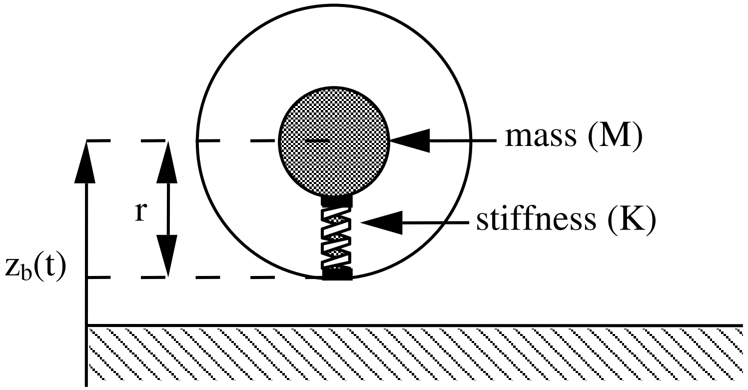
$$\dot{x}_3 = u$$

- An alternative representation using $x_1 = z_r - z_b$ i.e. new definition of x_1.

$$\dot{x}_1 = u - x_2$$

$$\dot{x}_2 = \frac{K}{M}x_1 + \frac{B_s}{M}(u - x_2)$$

Modelling Example 2



Looking back at the previous example and notes make sure you can obtain the following system equations;

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{K}{M}(r - x_1) - \frac{B}{M}x_2$$

Or (when not in contact with ground);

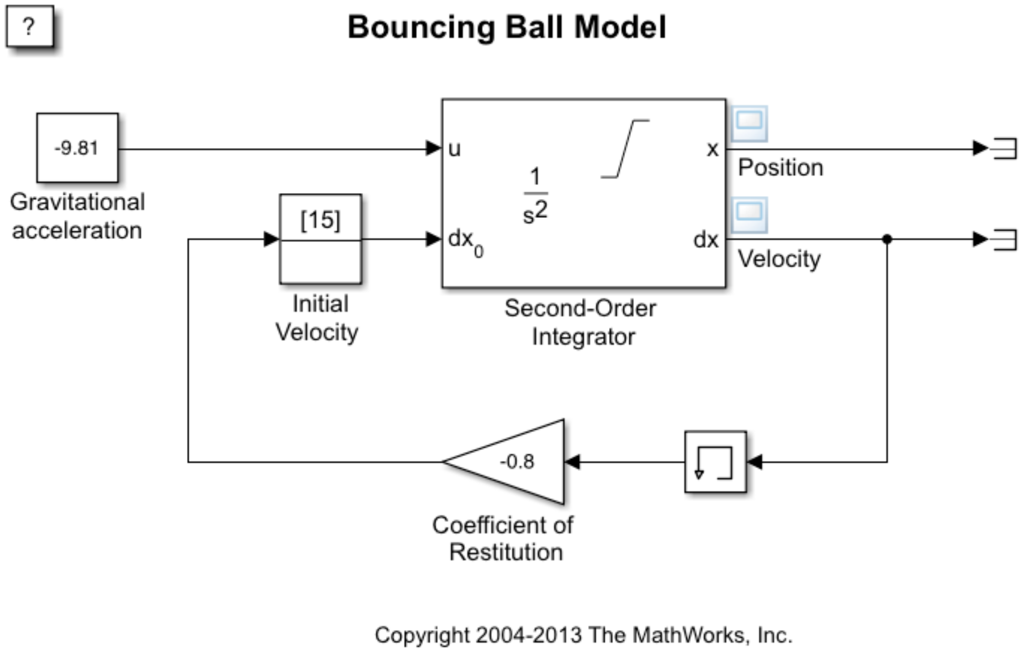
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g$$

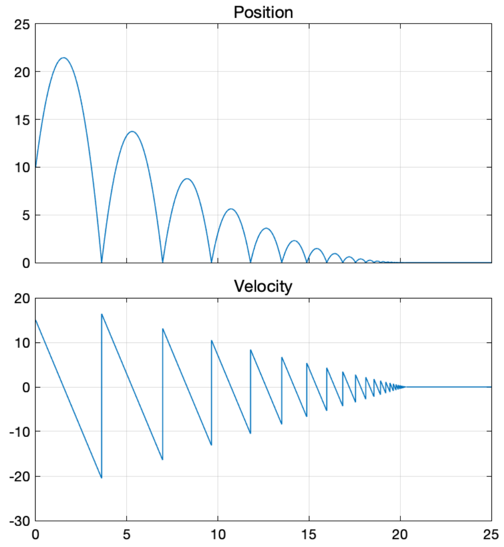
with states defined as follows;

$$x_1 = z_b \text{ and } x_2 = \dot{z}_b$$

Modelling Example 2

- From Simulink help open the bouncing ball model
- Familiarize yourself with the model implementation.
- Try changing the initial conditions and see how the model behavior changes.





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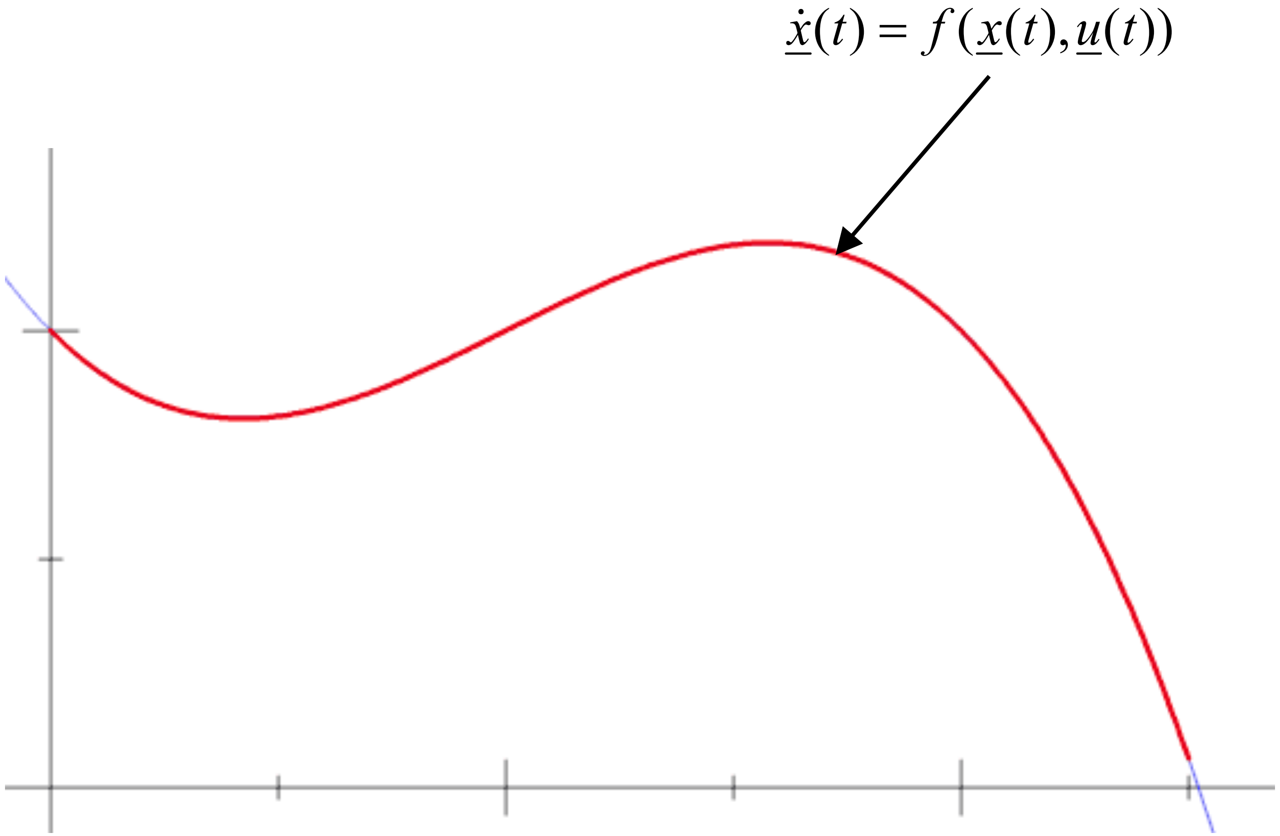
Numerical Integration of ODE's

- We often need to find the definite integral of some function (solution).

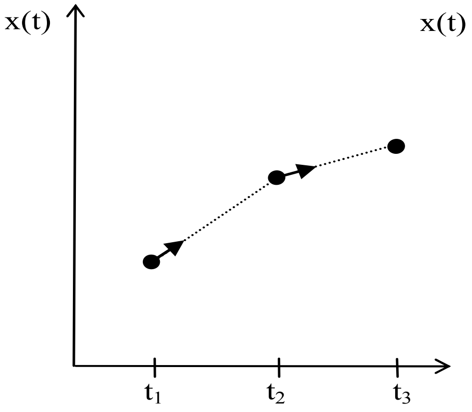
$$x(t) = \int_a^b \dot{x}\,(t)$$



Numerical Integration of ODE’s

Euler’s method

- Euler's forward method is a numerical integration technique that enables us to do this.
- $x(t+h)$ is evaluated using the gradient at t.
- h is the step size (small).
- Limitations;
  - Accuracy improved by reducing h i.e. the step size.
  - Large h can result in low accuracy and numerical instability.
  - Errors $\mathcal{O}(h^2)$



$$\underline{\dot{x}}(t) = f(\underline{x}(t), \underline{u}(t))$$

$$\underline{x}(t+h) = \underline{x}(t) + h\underline{\dot{x}}(t)$$

What is the consequence of reducing the size of h on the calculation?

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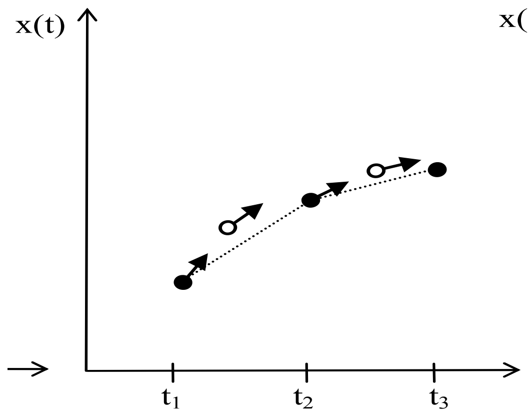
Numerical Integration of ODE's

Midpoint method

- The midpoint method evaluates $x(t+h)$ using the gradient at $t+h/2$
- Errors $\mathcal{O}(h^3)$



$$\underline{k}_1 = hf(\underline{x}(t),\ \underline{u}(t))$$

Value of function at h/2

$$\underline{k}_2 = hf(\underline{x}(t) + \underline{k}_1/2,\ \underline{u}(t+h/2))$$

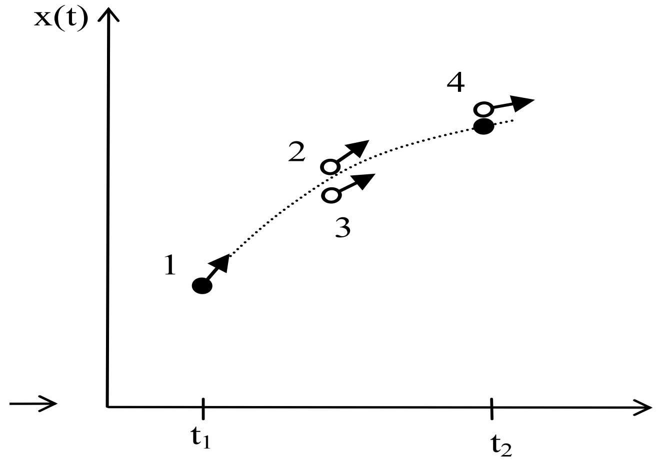
$$\underline{x}(t+h) = \underline{x}(t) + \underline{k}_2 + \mathrm{O}(h^3)$$

Value of input at h/2

Numerical Integration of ODE's

Runge-Kutta 4th order

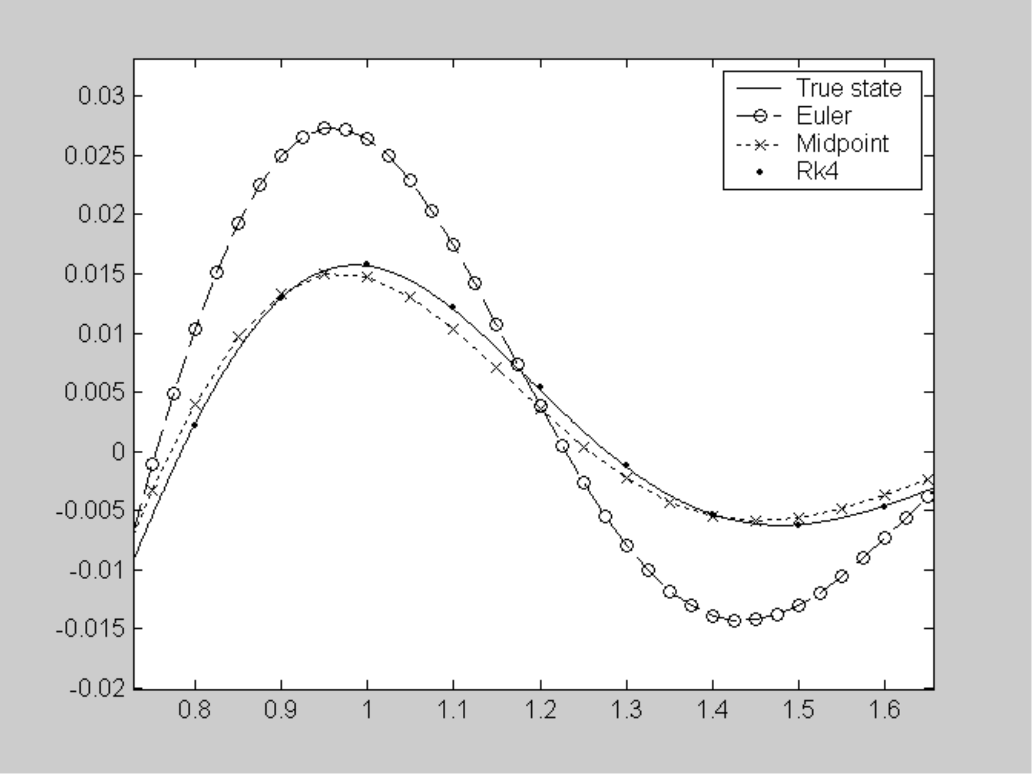
- RK4 evaluates $x(t+h)$ using the gradient at t, $t+h$ and two estimates at $t+h/2$
- $\mathcal{O}(h^5)$
- RK4 is the most used fixed step solver



$$\underline{k}_1 = hf(\underline{x}(t),\ \underline{u}(t))$$
$$\underline{k}_2 = hf(\underline{x}(t) + \underline{k}_1/2,\ \underline{u}(t+h/2))$$
$$\underline{k}_3 = hf(\underline{x}(t) + \underline{k}_2/2,\ \underline{u}(t+h/2))$$
$$\underline{k}_4 = hf(\underline{x}(t) + \underline{k}_3,\ \underline{u}(t+h))$$
$$\underline{x}(t+h) = \underline{x}(t) + \frac{\underline{k}_1}{6} + \frac{\underline{k}_2}{3} + \frac{\underline{k}_3}{3} + \frac{\underline{k}_4}{6} + \mathrm{O}(h^5)$$

Numerical Integration of ODE's

- Compare the results on the right for the three different integration algorithms.



In what circumstance would one opt for a lower accuracy method?

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Numerical Integration of ODE's

- Fixed and variable step 'solvers' are the two main categories.
- Variable step solvers change the step size during the solution.
- Example: bouncing ball. It is not always obvious what the solution is going to look like.
- Fixed step solvers are required for real time simulation.